

3. Elementary Analytical Methods

3.1. Binomial Theorem and Binomial Coefficients; Arithmetic and Geometric Progressions; Arithmetic, Geometric, Harmonic and Generalized Means

Binomial Theorem

3.1.1

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + b^n$$

(n a positive integer)

Binomial Coefficients (see chapter 24)

3.1.2

$$* \binom{n}{k} = {}_n C_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

3.1.3

$$\binom{n}{k} = \binom{n}{n-k} = (-1)^k \binom{k-n-1}{k}$$

3.1.4

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

3.1.5

$$\binom{n}{0} = \binom{n}{n} = 1$$

3.1.6

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

3.1.7

$$1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} = 0$$

Table of Binomial Coefficients $\binom{n}{k}$

3.1.8

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

For a more extensive table see chapter 24.

*See page II.

3.1.9

Sum of Arithmetic Progression to n Terms

$$a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$$

$$= na + \frac{1}{2} n(n-1)d = \frac{n}{2} (a+l),$$

last term in series = $l = a + (n-1)d$

Sum of Geometric Progression to n Terms

3.1.10

$$s_n = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n \rightarrow \infty} s_n = a/(1-r) \quad (-1 < r < 1)$$

Arithmetic Mean of n Quantities A

3.1.11

$$A = \frac{a_1 + a_2 + \dots + a_n}{n}$$

Geometric Mean of n Quantities G

3.1.12

$$G = (a_1 a_2 \dots a_n)^{1/n} \quad (a_k > 0, k=1, 2, \dots, n)$$

Harmonic Mean of n Quantities H

3.1.13

$$\frac{1}{H} = \frac{1}{n} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \quad (a_k > 0, k=1, 2, \dots, n)$$

Generalized Mean

3.1.14

$$M(t) = \left(\frac{1}{n} \sum_{k=1}^n a_k^t \right)^{1/t}$$

3.1.15

$$M(t) = 0 \quad (t < 0, \text{ some } a_k \text{ zero})$$

3.1.16

$$\lim_{t \rightarrow \infty} M(t) = \max. (a_1, a_2, \dots, a_n) = \max. a$$

3.1.17

$$\lim_{t \rightarrow -\infty} M(t) = \min. (a_1, a_2, \dots, a_n) = \min. a$$

3.1.18

$$\lim_{t \rightarrow 0} M(t) = G$$

3.1.19

$$M(1) = A$$

3.1.20

$$M(-1) = H$$

3.2. Inequalities

Relation Between Arithmetic, Geometric, Harmonic and Generalized Means

3.2.1

$$A \geq G \geq H, \text{ equality if and only if } a_1 = a_2 = \dots = a_n$$

3.2.2

$$\min. a < M(t) < \max. a$$