8. Legendre Functions

Mathematical Properties

Notation

The conventions used are $z=x+iy$, x, y real, and in particular, x always means a real number in the interval $-1\leq x\leq +1$ with $\cos\theta=x$ where θ is likewise a real number; n and m are positive integers or zero; ν and μ are unrestricted except where otherwise indicated.

Other notations are:

$$P^n(x) \text{ for } \frac{n!P_n(x)}{(2n-1)!!}$$

$$P_{nm}(x) \text{ for } (-1)^m P_n^m(x)$$

$$T_n^m(x) \text{ for } (-1)^m P_n^m(x)$$

$$\overline{P_n^m}(x) \text{ for } (-1)^m\sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}\,P_n^m(x)$$

$$\mathfrak{P}_\nu^\mu(z) \text{ for } P_\nu^\mu(z),\ \mathfrak{Q}_\nu^\mu(z) \text{ for } Q_\nu^\mu(z) \qquad (\mathscr{R}z>1)$$

$$\mathfrak{Q}_\nu^\mu(z) \text{ for } e^{\mu\pi i}Q_\nu^\mu(z)$$

$$Q_\nu^\mu(z) \text{ for } \frac{\sin(\nu+\mu)\pi}{\sin\nu\pi}\,Q_\nu^\mu(z)$$

Various other definitions of the functions occur as well as mixing of definitions.

8.1. Differential Equation

8.1.1

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left[\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right]w=0$$

Solutions

(Degree ν and order μ with singularities at $z=\pm1$, ∞ as ordinary branch points—μ, ν arbitrary complex constants.)

$P_\nu^\mu(z)$, $Q_\nu^\mu(z)$—Associated Legendre Functions (Spherical Harmonics) of the First and Second Kinds [2]

$$|\arg(z\pm1)|<\pi, \qquad |\arg z|<\pi$$

$$(z^2-1)^{\frac{1}{2}\mu}=(z-1)^{\frac{1}{2}\mu}(z+1)^{\frac{1}{2}\mu}$$

(For $P_\nu^\mu(z)$, $\mu=0$, Legendre polynomials, see chapter **22.**)

8.1.2

$$P_\nu^\mu(z)=\frac{1}{\Gamma(1-\mu)}\left[\frac{z+1}{z-1}\right]^{\frac{1}{2}\mu}F\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right) \qquad (|1-z|<2)$$

(For $F(a, b; c; z)$ see chapter **15.**)

8.1.3 $$Q_\nu^\mu(z)=e^{i\mu\pi}2^{-\nu-1}\pi^{\frac{1}{2}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}z^{-\nu-\mu-1}(z^2-1)^{\frac{1}{2}\mu}F\left(1+\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right) \qquad (|z|>1)$$

Alternate Forms

(Additional forms may be obtained by means of the transformation formulas of the hypergeometric function, see [8.1].)

8.1.4 $$P_\nu^\mu(z)=2^\mu\pi^{\frac{1}{2}}(z^2-1)^{-\frac{1}{2}\mu}\left\{\frac{F\left(-\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2};\frac{1}{2};z^2\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)}-2z\frac{F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2};\frac{3}{2};z^2\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}\right)}\right\} \qquad (|z^2|<1)$$

8.1.5 $$P_\nu^\mu(z)=\frac{2^{-\nu-1}\pi^{-\frac{1}{2}}\Gamma(-\frac{1}{2}-\nu)z^{-\nu+\mu-1}}{(z^2-1)^{\mu/2}\Gamma(-\nu-\mu)}F\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2};\nu+\frac{3}{2};z^{-2}\right)$$
$$+\frac{2^\nu\Gamma(\frac{1}{2}+\nu)z^{\nu+\mu}}{(z^2-1)^{\mu/2}\Gamma(1+\nu-\mu)}F\left(-\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2};\frac{1}{2}-\nu;z^{-2}\right) \qquad (|z^{-2}|<1)$$

8.1.6 $$e^{-i\mu\pi}Q_\nu^\mu(z)=\frac{\Gamma(1+\nu+\mu)\Gamma(-\mu)(z-1)^{\frac{1}{2}\mu}(z+1)^{-\frac{1}{2}\mu}}{2\Gamma(1+\nu-\mu)}F\left(-\nu,1+\nu;1+\mu;\frac{1-z}{2}\right)$$
$$+\frac{1}{2}\Gamma(\mu)(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu}F\left(-\nu,1+\nu;1-\mu;\frac{1-z}{2}\right) \qquad (|1-z|<2)$$ *

[2] The functions $Y_n^m(\theta,\varphi)=\left.\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}\right\}P_n^m(\cos\theta)$ called surface harmonics of the first kind, tesseral for $m<n$ and sectoral for $m=n$. With $0\leq\theta\leq\pi$, $0\leq\varphi\leq2\pi$, they are everywhere one valued and continuous functions on the surface of the unit sphere $x^2+y^2+z^2=1$ where $x=\sin\theta\cos\varphi$, $y=\sin\theta\sin\varphi$ and $z=\cos\theta$.

*See page II.