9.3.31 $J_\nu(\nu)\sim\frac{a}{\nu^{1/3}}\{1+\sum_{k=1}^{\infty}\frac{\alpha_k}{\nu^{2k}}\}-\frac{b}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{\beta_k}{\nu^{2k}}$

9.3.32 $Y_\nu(\nu)\sim-\frac{3^{1/2}a}{\nu^{1/3}}\{1+\sum_{k=1}^{\infty}\frac{\alpha_k}{\nu^{2k}}\}-\frac{3^{1/2}b}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{\beta_k}{\nu^{2k}}$

9.3.33 $J'_\nu(\nu)\sim\frac{b}{\nu^{2/3}}\{1+\sum_{k=1}^{\infty}\frac{\gamma_k}{\nu^{2k}}\}-\frac{a}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{\delta_k}{\nu^{2k}}$

9.3.34 $Y'_\nu(\nu)\sim\frac{3^{1/2}b}{\nu^{2/3}}\{1+\sum_{k=1}^{\infty}\frac{\gamma_k}{\nu^{2k}}\}+\frac{3^{1/2}a}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{\delta_k}{\nu^{2k}}$

where

$$a=\frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})}=.44730\ 73184,\qquad 3^{\frac{1}{2}}a=.77475\ 90021$$

$$b=\frac{2^{2/3}}{3^{1/3}\Gamma(\frac{1}{3})}=.41085\ 01939,\qquad 3^{\frac{1}{2}}b=.71161\ 34101$$

$$\alpha_0=1,\qquad \alpha_1=-\frac{1}{225}=-.00\dot{4},$$

$$\alpha_2=.00069\ 3735\ \ldots,\qquad \alpha_3=-.00035\ 38\ \ldots$$

$$\beta_0=\frac{1}{70}=.01428\ 57143\ \ldots,$$

$$\beta_1=-\frac{1213}{10\ 23750}=-.00118\ 48596\ \ldots,$$

$$\beta_2=.00043\ 78\ \ldots,\qquad \beta_3=-.00038\ \ldots$$

$$\gamma_0=1,\qquad \gamma_1=\frac{23}{3150}=.00730\ 15873\ \ldots,$$

$$\gamma_2=-.00093\ 7300\ \ldots,\qquad \gamma_3=.00044\ 40\ \ldots$$

$$\delta_0=\frac{1}{5},\quad \delta_1=-\frac{947}{3\ 46500}=-.00273\ 30447\ \ldots,$$

$$\delta_2=.00060\ 47\ \ldots,\qquad \delta_3=-.00038\ \ldots$$

Uniform Asymptotic Expansions

These are more powerful than the previous expansions of this section, save for **9.3.31** and **9.3.32**, but their coefficients are more complicated. They reduce to **9.3.31** and **9.3.32** when the argument equals the order.

9.3.35

$$J_\nu(\nu z)\sim\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\right\}$$

9.3.36

$$Y_\nu(\nu z)\sim-\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left\{\frac{\mathrm{Bi}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Bi}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\right\}$$

9.3.37

$$H_\nu^{(1)}(\nu z)\sim2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}\,(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{e^{2\pi i/3}\mathrm{Ai}'(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\right\}$$

When $\nu\to+\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z|\leq\pi-\epsilon$, where ϵ is an arbitrary positive number. The corresponding expansion for $H_\nu^{(2)}(\nu z)$ is obtained by changing the sign of i in **9.3.37.**

Here

9.3.38

$$\frac{2}{3}\zeta^{3/2}=\int_z^1\frac{\sqrt{1-t^2}}{t}\,dt=\ln\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}$$

equivalently,

9.3.39

$$\frac{2}{3}(-\zeta)^{3/2}=\int_1^z\frac{\sqrt{t^2-1}}{t}\,dt=\sqrt{z^2-1}-\arccos\left(\frac{1}{z}\right)$$

the branches being chosen so that ζ is real when z is positive. The coefficients are given by

9.3.40

$$a_k(\zeta)=\sum_{s=0}^{2k}\mu_s\zeta^{-3s/2}u_{2k-s}\{(1-z^2)^{-\frac{1}{2}}\}$$

$$b_k(\zeta)=-\zeta^{-\frac{1}{2}}\sum_{s=0}^{2k+1}\lambda_s\zeta^{-3s/2}u_{2k-s+1}\{(1-z^2)^{-\frac{1}{2}}\}$$

where u_k is given by **9.3.9** and **9.3.10,** $\lambda_0=\mu_0=1$ and

9.3.41

$$\lambda_s=\frac{(2s+1)(2s+3)\ldots(6s-1)}{s!(144)^s},\quad \mu_s=-\frac{6s+1}{6s-1}\lambda_s$$

Thus $a_0(\zeta)=1$,

9.3.42

$$b_0(\zeta)=-\frac{5}{48\zeta^2}+\frac{1}{\zeta^{\frac{1}{2}}}\left\{\frac{5}{24(1-z^2)^{3/2}}-\frac{1}{8(1-z^2)^{\frac{1}{2}}}\right\}$$

$$=-\frac{5}{48\zeta^2}+\frac{1}{(-\zeta)^{\frac{1}{2}}}\left\{\frac{5}{24(z^2-1)^{3/2}}+\frac{1}{8(z^2-1)^{\frac{1}{2}}}\right\}$$

Tables of the early coefficients are given below. For more extensive tables of the coefficients and for bounds on the remainder terms in **9.3.35** and **9.3.36** see [9.38].