10.2. Modified Spherical Bessel Functions

Definitions

Differential Equation

10.2.1

$$z^2w''+2zw'-[z^2+n(n+1)]w=0 \qquad (n=0, \pm 1, \pm 2, \ldots)$$

Particular solutions are the *Modified Spherical Bessel functions of the first kind,*

10.2.2

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=e^{-n\pi i/2}j_n(ze^{\pi i/2}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)$$
$$=e^{3n\pi i/2}j_n(ze^{-3\pi i/2}) \qquad (\tfrac{1}{2}\pi<\arg z\leq\pi)$$

of the second kind,

10.2.3

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=e^{3(n+1)\pi i/2}y_n(ze^{\pi i/2}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)$$
$$=e^{-(n+1)\pi i/2}y_n(ze^{-3\pi i/2}) \qquad (\tfrac{1}{2}\pi<\arg z\leq\pi)$$

of the third kind,

10.2.4

$$\sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)=\tfrac{1}{2}\pi(-1)^{n+1}\sqrt{\tfrac{1}{2}\pi/z}\,[I_{n+\frac{1}{2}}(z)-I_{-n-\frac{1}{2}}(z)]$$

The pairs

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)$$

and

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)$$

are linearly independent solutions for every n.

Most properties of the Modified Spherical Bessel functions can be derived from those of the Spherical Bessel functions by use of the above relations.

Ascending Series

10.2.5

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=\frac{z^n}{1\cdot 3\cdot 5\ldots(2n+1)}\left\{1+\frac{\tfrac{1}{2}z^2}{1!(2n+3)}+\frac{(\tfrac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}+\ldots\right\}$$

10.2.6

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{(-1)^n z^{n+1}}\left\{1+\frac{\tfrac{1}{2}z^2}{1!(1-2n)}+\frac{(\tfrac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}+\ldots\right\} \qquad (n=0, 1, 2, \ldots)$$

Wronskians

10.2.7

$$W\{\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)\}=(-1)^{n+1}z^{-2}$$

10.2.8

$$W\{\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}\,K_{n+\frac{1}{2}}(z)\}=-\tfrac{1}{2}\pi z^{-2}$$

Representations by Elementary Functions

10.2.9

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z-(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.10

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n-\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z+(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.11

$$R(n+\tfrac{1}{2},z)=1+\frac{(n+1)!}{1!\Gamma(n)}(2z)^{-1}+\frac{(n+2)!}{2!\Gamma(n-1)}(2z)^{-2}+\ldots$$
$$=\sum_0^n(n+\tfrac{1}{2},k)(2z)^{-k} \qquad (n=0, 1, 2, \ldots)$$

(See **10.1.9.**)

10.2.12

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{n+\frac{1}{2}}(z)=g_n(z)\sinh z+g_{-n-1}(z)\cosh z$$
$$g_0(z)=z^{-1},\ g_1(z)=-z^{-2}$$
$$g_{n-1}(z)-g_{n+1}(z)=(2n+1)z^{-1}g_n(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

The Functions $\sqrt{\tfrac{1}{2}\pi/z}\,I_{\pm(n+\frac{1}{2})}(z)$, $n=0, 1, 2 \cdot$

10.2.13

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{1/2}(z)=\frac{\sinh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{3/2}(z)=-\frac{\sinh z}{z^2}+\frac{\cosh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{5/2}(z)=\left(\frac{3}{z^3}+\frac{1}{z}\right)\sinh z-\frac{3}{z^2}\cosh z$$

10.2.14

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-1/2}(z)=\frac{\cosh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-3/2}(z)=\frac{\sinh z}{z}-\frac{\cosh z}{z^2}$$

$$\sqrt{\tfrac{1}{2}\pi/z}\,I_{-5/2}(z)=-\frac{3}{z^2}\sinh z+\left(\frac{3}{z^3}+\frac{1}{z}\right)\cosh z$$

*See page II.