Example 2. Compute $j_{15}(x)$ for $x=24.6$.

Interpolation in **Table 10.3** yields for $x=24.6$

$$x^{-21}e^{x^2/86}j_{21}(x)=(-28)3.934616$$
$$x^{-20}e^{x^2/82}j_{20}(x)=(-27)9.48683$$

whence

$$j_{21}(24.6)=.05604\ 29,\ j_{20}(24.6)=.03896\ 98.$$

From the recurrence relation **10.1.19** there results

$$j_{19}(24.6)=\ \ .00890\ 67660 \qquad [.00890\ 70]$$
$$j_{18}(24.6)=-.02484\ 93173 \qquad [-.02485\ 90]$$
$$j_{17}(24.6)=-.04628\ 17554 \qquad [-.04628\ 16]$$
$$j_{16}(24.6)=-.04099\ 87086 \qquad [-.04099\ 88]$$
$$j_{15}(24.6)=-.00871\ 65122 \qquad [-.00871\ 67]$$

For comparison, the correct values are shown in brackets.

To compute $j_{15}(x)$ for $x=24.6$ by Miller's device, take, for example, $N=39$ and assume $F_{40}=0$, $F_{39}=1$. Using **10.1.19** with decreasing N, i.e., $F_{N-1}=[(2N+1)/x]F_N-F_{N+1}$, $N=39, 38, \ldots, 1, 0$, generate the sequence $F_{38}, F_{37}, \ldots, F_1, F_0$, compute from **Table 4.6,** $j_0(24.6)=(\sin 24.6)/24.6=-.02064\ 620296$, and obtain the factor of proportionality

$$p=j_0(24.6)/F_0=.00000\ 03839\ 17642.$$

The value pF_{15} equals $j_{15}(24.6)$ to 8 decimals. The final part of the computations is shown in the following table, in which the correct values are given for comparison.

N	F_N	pF_N	$j_N(24.6)$
15	−22704. 71107	−. 00871 67391	−. 00871 674
14	+78178. 88236	+. 03001 42522	+. 03001 425
13	+114866. 80811	+. 04409 93941	+. 04409 939
12	+47894. 44353	+. 01838 75218	+. 01838 752
11	−66193. 59317	−. 02541 28882	−. 02541 289
10	−109782. 76234	−. 04214 75392	−. 04214 754
9	−27523. 39903	−. 01056 67185	−. 01056 672
8	+88524. 85252	+. 03398 62526	+. 03398 625
7	+88699. 11017	+. 03405 31532	+. 03405 315
6	−34440. 02929	−. 01322 21348	−. 01322 213
5	−106899. 12565	−. 04104 04602	−. 04104 046
4	−13360. 39272	−. 00512 92905	−. 00512 929
3	+102011. 17704	+. 03916 38905	+. 03916 389
2	+42387. 96341	+. 01627 34870	+. 01627 349
1	−93395. 73728	−. 03585 62712	−. 03585 627
0	−53777. 68747	−. 02064 62030	−. 02064 620

It may be observed that the normalization of the sequence $F_N, F_{N-1}, \ldots, F_0$ can also be obtained from formula **10.1.50** by computing the sum $\sigma=\sum_0^\infty (2k+1)F_k^2$ and finding $p=1/\sqrt{\sigma}$. This yields, in the case of the example, $p=1/\sqrt{\sigma}=.00000\ 03839\ 177$.

Modified Spherical Bessel Functions

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$, $n=0, 1, 2, \ldots$ for values of x outside the range of **Table 10.8,** use formulas **10.2.13, 10.2.14** together with **10.2.4** and obtain values for the hyperbolic and exponential functions from **Tables 4.4** and **4.15.** In those cases when $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$ and $\sqrt{\frac{1}{2}\pi/x}I_{-n-\frac{1}{2}}(x)$ are nearly equal, i.e., when x is sufficiently large, compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ from formula **10.2.15,** for which the coefficients $(n+\frac{1}{2}, k)$ are given in **10.1.9.**

Example 3. Compute $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{5/2}(x)$ for $x=16.2$.

From **10.2.13,** $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)=(3+x^2)\sinh x/x^3-3\cosh x/x^2$; from **Table 4.4,** $\cosh 16.2=(6)5.4267\ 59950$ and this equals the value of $\sinh 16.2$ to the same number of significant figures. Hence

$$\begin{aligned}\sqrt{\tfrac{1}{2}\pi/16.2}I_{5/2}(16.2)&=(.06243\ 402371\\&\quad-.01143\ 118427)[(6)5.4267\ 59950]\\&=338814.4594-62034.29298\\&=276780.1664.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/16.2}K_{5/2}(16.2)$ use **10.2.17** and obtain

$$\begin{aligned}\sqrt{\tfrac{1}{2}\pi/16.2}K_{5/2}(16.2)&=\pi e^{-16.2}\left[\frac{1}{32.4}+\frac{6}{(32.4)^2}+\frac{12}{(32.4)^3}\right]\\&=(-7)2.8945\ 38069[.036932\ 60400]\\&=(-8)1.0690\ 28283.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $3\leq n\leq 8$, for a value of x within the range of **Table 10.9,** obtain from **Table 10.9,** $\sqrt{\frac{1}{2}\pi/x}I_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}I_{21/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation **10.2.18** for decreasing n.

To compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ for some integer n outside the range of **Table 10.9,** obtain from **10.2.15** or from **Table 10.8,** $\sqrt{\frac{1}{2}\pi/x}K_{\frac{1}{2}}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{3/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation **10.2.18** for increasing n. If x lies within the range of **Table 10.9** and $n>10$, the recurrence may be started with $\sqrt{\frac{1}{2}\pi/x}K_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{21/2}(x)$ obtained from **Table 10.9.**

Example 4. Compute $\sqrt{\frac{1}{2}\pi/x}K_{11/2}(x)$ for $x=3.6$. Obtain from **Table 10.8** for $x=3.6$

$$\sqrt{\tfrac{1}{2}\pi/x}K_{1/2}(x)=.01192\ 222$$

$$\sqrt{\tfrac{1}{2}\pi/x}K_{3/2}(x)=.01523\ 3952$$