14.6.15 $\rho=2\eta>>0$

$$\frac{F_0}{G_0/\sqrt{3}}\sim\frac{\Gamma(1/3)}{2\sqrt{\pi}}\left(\frac{2\eta}{3}\right)^{1/6}$$

$$\frac{F_0'}{-G_0'/\sqrt{3}}\sim\frac{\Gamma(2/3)}{2\sqrt{\pi}(2\eta/3)^{1/6}}$$

14.6.16 $\eta\rightarrow\infty$

$$\sigma_0(\eta)\sim[\frac{\pi}{4}+\eta(\ln\eta-1)]$$

$$C_0(\eta)\sim(2\pi\eta)^{1/2}e^{-\pi\eta},$$

(Equality to 8S for $\eta>3$.)

14.6.17 $\eta\rightarrow 0$

$$\sigma_0(\eta)\sim-\gamma\eta\qquad(\gamma=\text{Euler's constant})$$

$$C_L(\eta)\sim\frac{2^L L!}{(2L+1)!}$$

14.6.18 $L\rightarrow\infty$

$$C_L(\eta)\sim\frac{2^L L!}{(2L+1)!}e^{-\pi\eta/2}$$

Numerical Methods

14.7. Use and Extension of the Tables

In general the tables as presented are not simply interpolable. However, values for $L>0$ may be obtained with the help of the recurrence relations. The values of $G_L(\eta,\rho)$ may be obtained by applying the recurrence relations in increasing order of L. Forward recurrence may be used for $F_L(\eta,\rho)$ as long as the instability does not produce errors in excess of the accuracy needed. In this case the backwards recurrence scheme (see **Example 1**) should be used.

Example 1. Compute $F_L(\eta,\rho)$ and $F_L'(\eta,\rho)$ for $\eta=2$, $\rho=5$, $L=0(1)5$. Starting with $F_{10}^*=1$, $F_{11}^*=0$, where $F_L^*=cF_L$, we compute from **14.2.3** in decreasing order of L:

L	(1) F_L^*	(2) F_L	(3) F_L	(4) F_L'
11	0.			
10	1.			
9	4.49284			
8	17.5225			
7	61.3603			
6	191.238			
5	523.472	.090791	.091	.1043
4	1238.53	.21481	.215	.2030
3	2486.72	.43130	.4313	.3205
2	4158.46	.72124	.72125	.3952
1	5727.97	.99346	.99347	.3709
0	6591.81	1.1433	1.1433	.29380

$F_0/F_0^*=1.7344\times10^{-4}=c^{-1}$.

The values in the second column are obtained from those in the first by multiplying by the normalization constant, F_0/F_0^* where F_0 is the known value obtained from **Table 14.1.**

Repetition starting with $F_{15}^*=1$ and $F_{16}^*=0$ yields the same results.

In column 3, the results have been given when **14.2.3** is used in increasing order of L.

F_L' (column 4) follows from **14.2.2.**

Example 2. Compute $G_L(\eta,\rho)$ and $G_L'(\eta,\rho)$ for $\eta=2$, $\rho=5$, $L=1(1)5$.

Using **14.2.2** and $G_0(2,5)=.79445$, $G_0'=-.67049$ from **Table 14.1** we find $G_1(2,5)=1.0815$. Then by forward recurrence using **14.2.3** we find:

L	G_L	$-G'_L$ *
1	1.0815	.60286
2	1.4969	.56619
3	2.0487	.79597
4	3.0941	1.7318
5	5.6298	4.5493

The values of G_L' are obtained with **14.2.1.**

Example 3. Compute $G_0(\eta,\rho)$ for $\eta=2$, $\rho=2.5$.

From **Table 14.1**, $G_0(2,2)=3.5124$, $G_0'(2,2)=-2.5554$. Successive differentiation of **14.1.1** for $L=0$ gives

$$\rho\frac{d^{k+2}w}{d\rho^{k+2}}=(2\eta-\rho)\frac{d^kw}{d\rho^k}-k\{\frac{d^{k+1}w}{d\rho^{k+1}}+\frac{d^{k-1}w}{d\rho^{k-1}}\}$$

Taylor's expansion is $w(\rho+\Delta\rho)=w(\rho)+(\Delta\rho)w'+\frac{(\Delta\rho)^2}{2!}w''+\ldots.$ With $w=G_0(\eta,\rho)$ and $\Delta\rho=.5$ we get:

k	$\frac{d^kG_0}{d\rho^k}$	$\frac{(\Delta\rho)^k}{k!}\frac{d^kG_0}{d\rho^k}$
0	3.5124	3.5124
1	−2.5554	−1.2777
2	3.5124	.43905
3	−6.0678	−.12641
4	12.136	.03160
5	−29.540	−.00769
6	83.352	.00181
7	−268.26	−.00042

$G_0(2, 2.5)=2.5726$

As a check the result is obtained with $\eta=2$, $\rho=3$, $\Delta\rho=-.5$. The derivative $G_0'(\eta,\rho)$ may be obtained using Taylor's formula with $w=G_0'(\eta,\rho)$.

*See page II.