15. Hypergeometric Functions

Mathematical Properties

15.1. Gauss Series, Special Elementary Cases, Special Values of the Argument

Gauss Series

The circle of convergence of the Gauss hypergeometric series

15.1.1

$$F(a,b;c;z)={}_2F_1(a,b;c;z)$$

$$=F(b,a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$

is the unit circle $|z|=1$. The behavior of this series on its circle of convergence is:

(a) Divergence when $\mathscr{R}(c-a-b)\leqq -1$.

(b) Absolute convergence when $\mathscr{R}(c-a-b)>0$.

(c) Conditional convergence when $-1<\mathscr{R}(c-a-b)\leqq 0$; the point $z=1$ is excluded. The Gauss series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n=0, 1, 2, \ldots)$. (For these cases see also **15.4.**) The series **15.1.1** is not defined when c is equal to $-m$, $(m=0, 1, 2, \ldots)$, provided a or b is not a negative integer n with $n<m$. For $c=-m$

15.1.2

$$\lim_{c\to -m}\frac{1}{\Gamma(c)}F(a,b;c;z)=$$

$$\frac{(a)_{m+1}(b)_{m+1}}{(m+1)!}z^{m+1}F(a+m+1,b+m+1;m+2;z)$$

Special Elementary Cases of Gauss Series

(For cases involving higher functions see **15.4.**)

15.1.3 $\quad F(1,1;2;z)=-z^{-1}\ln(1-z)$ *

15.1.4 $\quad F(\tfrac{1}{2},1;\tfrac{3}{2};z^2)=\tfrac{1}{2}z^{-1}\ln\left(\frac{1+z}{1-z}\right)$

15.1.5 $\quad F(\tfrac{1}{2},1;\tfrac{3}{2};-z^2)=z^{-1}\arctan z$

15.1.6

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2)=(1-z^2)^{\frac{1}{2}}F(1,1;\tfrac{3}{2};z^2)=z^{-1}\arcsin z$$

15.1.7

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-z^2)=(1+z^2)^{\frac{1}{2}}F(1,1;\tfrac{3}{2};-z^2)$$
$$=z^{-1}\ln[z+(1+z^2)^{\frac{1}{2}}]$$

15.1.8 $\quad F(a,b;b;z)=(1-z)^{-a}$

15.1.9 $\quad F(a,\tfrac{1}{2}+a;\tfrac{1}{2};z^2)=\tfrac{1}{2}[(1+z)^{-2a}+(1-z)^{-2a}]$

15.1.10

$$F(a,\tfrac{1}{2}+a;\tfrac{3}{2};z^2)=$$
$$\tfrac{1}{2}z^{-1}(1-2a)^{-1}[(1+z)^{1-2a}-(1-z)^{1-2a}]$$

15.1.11

$$F(-a,a;\tfrac{1}{2};-z^2)=\tfrac{1}{2}\{[(1+z^2)^{\frac{1}{2}}+z]^{2a}+[(1+z^2)^{\frac{1}{2}}-z]^{2a}\}$$

15.1.12

$$F(a,1-a;\tfrac{1}{2};-z^2)=$$
$$\tfrac{1}{2}(1+z^2)^{-\frac{1}{2}}\{[(1+z^2)^{\frac{1}{2}}+z]^{2a-1}+[(1+z^2)^{\frac{1}{2}}-z]^{2a-1}\}$$

15.1.13

$$F(a,\tfrac{1}{2}+a;1+2a;z)=2^{2a}[1+(1-z)^{\frac{1}{2}}]^{-2a}$$
$$=(1-z)^{\frac{1}{2}}F(1+a,\tfrac{1}{2}+a;1+2a;z)$$

15.1.14

$$F(a,\tfrac{1}{2}+a;2a;z)=2^{2a-1}(1-z)^{-\frac{1}{2}}[1+(1-z)^{\frac{1}{2}}]^{1-2a}$$

15.1.15 $\quad F(a,1-a;\tfrac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-1)z]}{(2a-1)\sin z}$

15.1.16 $\quad F(a,2-a;\tfrac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-2)z]}{(a-1)\sin(2z)}$

15.1.17 $\quad F(-a,a;\tfrac{1}{2};\sin^2 z)=\cos(2az)$

15.1.18 $\quad F(a,1-a;\tfrac{1}{2};\sin^2 z)=\dfrac{\cos[(2a-1)z]}{\cos z}$

15.1.19 $\quad F(a,\tfrac{1}{2}+a;\tfrac{1}{2};-\tan^2 z)=\cos^{2a}z\cos(2az)$

Special Values of the Argument

15.1.20

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$(c\neq 0,-1,-2,\ldots,\ \mathscr{R}(c-a-b)>0)$$

*See page II.