known multiple of the solution desired. The computation is then carried back until a value of a with $|a|\leq 1$ is reached, when the precise multiple that we have of the desired solution may be determined and hence removed throughout. Compare also **9.12, Example 1.**

Example 1. Evaluate $U(a, 5)$ for $a=5, 6, 7, \ldots$, using **19.6.4.**

$$(a+\tfrac{1}{2})U(a+1, x)+xU(a, x)-U(a-1, x)=0$$

a	Forward Recurrence	Backward Recurrence	Final Values
3	(−6) 5.2847*	(12) 1.59035	(−6) 5.2847**
4	(−7) 9.172*	(11) 2.76028	(−7) 9.1724
5	(−7) 1.5527	(10) 4.67131	(−7) 1.55227
6	(−8) 2.5609	(9) 7.72041	(−8) 2.5655
7	(−9) 4.1885	(9) 1.24785	(−9) 4.1466
8	(−10) 6.2220	(8) 1.97488	(−10) 6.5625
9	(−10) +1.2676	(7) 3.06369	(−10) 1.01806
10	(−11) −0.1221	(6) 4.66352	(−11) 1.5497
11	(−11) +1.2654	(0) 697082	(−12) 2.3164
12	(−12) −5.6079	102444	(−13) 3.404
13	(−12) +3.2555	14789	(−14) 4.91
14		2111	(−15) 7.01
15		292	(−16) 9.7
16		42	
17		5	
18		1+	
19		0+	

*From tables. +Starting values.

**This value was used to obtain the constant multiplier $\frac{d}{k^*}=\frac{(-6)5.2847}{(12)1.59035}=(-18)3.32298$ for converting the previous column into this one.

The second column shows forward recurrence starting with values at $a=3, 4$ from **Table 19.1.** Backward recurrence starts with values 0 and 1 at $a=19$ and 18, containing a multiple $kU(a, 5)$ and a subsequently negligible multiple of the other solution $\Gamma(\frac{1}{2}-a)V(a, 5)$. Rounding errors convert $kU(a, x)$ into $k^*U(a, x)$ *without affecting the values in the last column.* The value of $1/k^*$ is identified from the known value of $U(3, 5)$, and used to obtain the final column by multiplying throughout by $1/k^*$. The improvement in $U(5, 5)$ is evident by comparison with **Table 19.1.**

Derivatives. These are not tabulated here. Since the functions $U(a, x)$, $V(a, x)$ and $W(a, x)$ satisfy differential equations, values of derivatives are often required.

For all these functions the equation is second order with first derivative absent, so that *second derivatives* may be readily obtained from function values by use of the differential equation.

First derivatives can be obtained for $U(a, x)$ and $V(a, x)$ by applying the appropriate recurrence relations **19.6.1–2.** If less accuracy is needed they can be found by use of mean central differences of $U(a, x)$, $V(a, x)$ and also of $W(a, x)$ with the formula

$$hu'=h\frac{du}{dx}=\mu\delta u-\tfrac{1}{6}\mu\delta^3 u+\tfrac{1}{30}\mu\delta^5 u-\ldots$$

using $h=.1$; this usually gives a 3- or 4-figure value of du/dx.

If greater accuracy is needed for $dW(a, x)/dx$ it may be obtained by evaluating d^2W/dx^2 with the help of the differential equation satisfied by W and integrating this second derivative numerically. This requires one accurate value of dW/dx to start off the integration; we describe two methods for obtaining this, both making use of the difference between two fairly widely separated values of W, for example, separated by 5 or 10 tabular intervals.

(i) Write f_r, f'_r, f''_r for $W(a, x_0+rh)$ and its first two derivatives, then f'_0 may be found from

$$hf'_0=\frac{1}{2n}(f_n-f_{-n})-\frac{h^2}{2n}\sum_1^{n-1}(n-r)(f''_r-f''_{-r})$$
$$-\frac{h^2}{2n}\{\tfrac{1}{12}-\tfrac{1}{240}\delta^2+\tfrac{31}{60480}\delta^4-\ldots\}(f''_n-f''_{-n})$$
$$-h^2\{\tfrac{1}{12}\mu\delta-\tfrac{11}{720}\mu\delta^3+\tfrac{191}{60480}\mu\delta^5-\ldots\}f''_0$$

(ii) Consider a solution y of the differential equation for $W(a, x)$, namely $y''=(-\frac{1}{4}x^2+a)y$. If we are given values y and y' at a particular $x=x_0$ and write $T_n=H^n y^{(n)}/n!$, $T_{-1}=T_{-2}=0$, then we may compute T_2, T_3, T_4, . . . in succession by use of the recurrence relation obtained from the differential equation,

$$T_{n+2}=\frac{H^2}{(n+1)(n+2)}[(-\tfrac{1}{4}x_0^2+a)T_n-\tfrac{1}{2}Hx_0T_{n-1}-\tfrac{1}{4}H^2T_{n-2}]$$

These are computed, to a fixed number of decimals until they become negligible, thus giving

$$y(x_0\pm H)=T_0\pm T_1+T_2\pm T_3+\ldots$$

This may be applied, with $H=rh$, h being the tabular interval, and r a small integer, say $r=5$, to the solutions $y=y_1$, $y=y_2$ having

$$y_1(x_0)=W(a, x_0) \qquad y'_1(x_0)=W^{*\prime}(a, x_0)$$
$$y_2(x_0)=0 \qquad y'_2(x_0)=1$$

in which $W^{*\prime}(a, x_0)$ is an approximation to $W'(a, x_0)$, not necessarily a good one; it may be