4.2.14 $\arg(e^z)=y$

4.2.15 $a^{z_1}a^{z_2}=a^{z_1+z_2}$

4.2.16 $a^z b^z=(ab)^z \qquad (-\pi<\arg a+\arg b\leq\pi)$

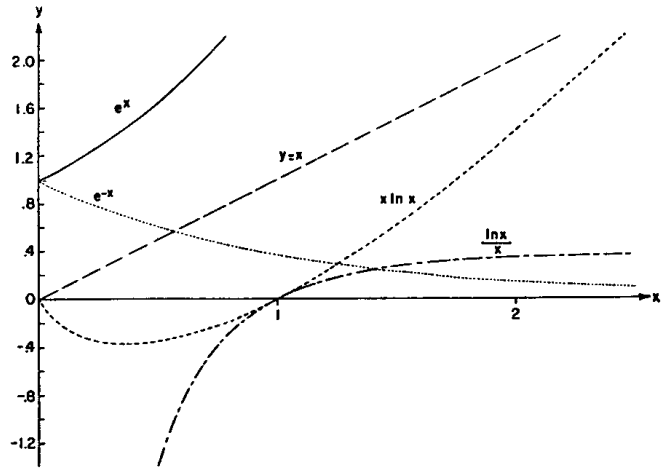

FIGURE 4.2. *Logarithmic and exponential functions.*

Periodic Property

4.2.17 $e^{z+2\pi ki}=e^z \qquad (k \text{ any integer})$

Exponential Identities

4.2.18 $e^{z_1}e^{z_2}=e^{z_1+z_2}$

4.2.19 $(e^{z_1})^{z_2}=e^{z_1 z_2} \qquad (-\pi<\mathscr{I}z_1\leq\pi)$

The restriction $(-\pi<\mathscr{I}z_1\leq\pi)$ can be removed if z_2 is an integer.

Limiting Values

4.2.20

$$\lim_{|z|\to\infty} z^\alpha e^{-z}=0 \qquad (|\arg z|\leq\tfrac{1}{2}\pi-\epsilon<\tfrac{1}{2}\pi, \quad \alpha \text{ constant})$$

4.2.21

$$\lim_{m\to\infty}\left(1+\frac{z}{m}\right)^m=e^z$$

Special Values (see chapter 1)

4.2.22 $e=2.71828\ 18284\ldots$

4.2.23 $e^0=1$

4.2.24 $e^\infty=\infty$

4.2.25 $e^{-\infty}=0$

4.2.26 $e^{\pm\pi i}=-1$

4.2.27 $e^{\pm\frac{\pi i}{2}}=\pm i$

4.2.28 $e^{2\pi ki}=1 \qquad (k \text{ any integer})$

Exponential Inequalities

If x is real and different from zero

4.2.29 $e^{-\frac{x}{1-x}}<1-x<e^{-x} \qquad (x<1)$

4.2.30 $e^x>1+x$

4.2.31 $e^x<\dfrac{1}{1-x} \qquad (x<1)$

4.2.32 $\dfrac{x}{1+x}<(1-e^{-x})<x \qquad (x>-1)$

4.2.33 $x<(e^x-1)<\dfrac{x}{1-x} \qquad (x<1)$

4.2.34 $1+x>e^{\frac{x}{1+x}} \qquad (x>-1)$

4.2.35 $e^x>1+\dfrac{x^n}{n!} \qquad (n>0, \quad x>0)$

4.2.36 $e^x>\left(1+\dfrac{x}{y}\right)^y>e^{\frac{xy}{x+y}} \qquad (x>0, \quad y>0)$

4.2.37 $e^{-x}<1-\dfrac{x}{2} \qquad (0<x\leq 1.5936)$

4.2.38 $\dfrac{1}{4}|z|<|e^z-1|<\dfrac{7}{4}|z| \qquad (0<|z|<1)$

4.2.39 $|e^z-1|\leq e^{|z|}-1\leq|z|e^{|z|} \qquad (\text{all } z)$

Continued Fractions

4.2.40

$$e^z=\cfrac{1}{1-}\,\cfrac{z}{1+}\,\cfrac{z}{2-}\,\cfrac{z}{3+}\,\cfrac{z}{2-}\,\cfrac{z}{5+}\,\cfrac{z}{2-}\cdots \qquad (|z|<\infty)$$

$$=1+\cfrac{z}{1-}\,\cfrac{z}{2+}\,\cfrac{z}{3-}\,\cfrac{z}{2+}\,\cfrac{z}{5-}\,\cfrac{z}{2+}\,\cfrac{z}{7-}\cdots \qquad (|z|<\infty)$$

$$=1+\cfrac{z}{(1-z/2)+}\,\cfrac{z^2/4\cdot 3}{1+}\,\cfrac{z^2/4\cdot 15}{1+}\,\cfrac{z^2/4\cdot 35}{1+}\cdots\cfrac{z^2/4(4n^2-1)}{1+}\cdots \qquad (|z|<\infty)$$

4.2.41

$$e^z-e_{n-1}(z)=\cfrac{z^n}{n!-}\,\cfrac{n!z}{(n+1)+}\,\cfrac{z}{(n+2)-}\,\cfrac{(n+1)z}{(n+3)+}\,\cfrac{2z}{(n+4)-}\,\cfrac{(n+2)z}{(n+5)+}\,\cfrac{3z}{(n+6)-}\cdots \qquad (|z|<\infty)$$

(For $e_n(z)$ see **6.5.11**)