following:

(a) For a fixed real q, characteristic values a_r and b_r are real and distinct, if $q\neq 0$; $a_0<b_1<a_1<b_2<a_2<\ldots$, $q>0$ and $a_r(q)$, $b_r(q)$ approach r^2 as q approaches zero.

(b) A solution of **20.1.1** associated with a_r or b_r has r zeros in the interval $0\leq z<\pi$, (q real).

(c) The form of **20.2.21** and **20.2.23** shows that if a_{2r} is a root of **20.2.21** and q is different from zero, then a_{2r} cannot be a root of **20.2.23**; similarly, no root of **20.2.22** can be a root of **20.2.24** if $q\neq 0$. It may be shown from other considerations that for a given point (a, q) there can be at most one periodic solution of period π or 2π if $q\neq 0$. This no longer holds for solutions of period $s\pi$, $s\geq 3$; for these all solutions are periodic, if one is.

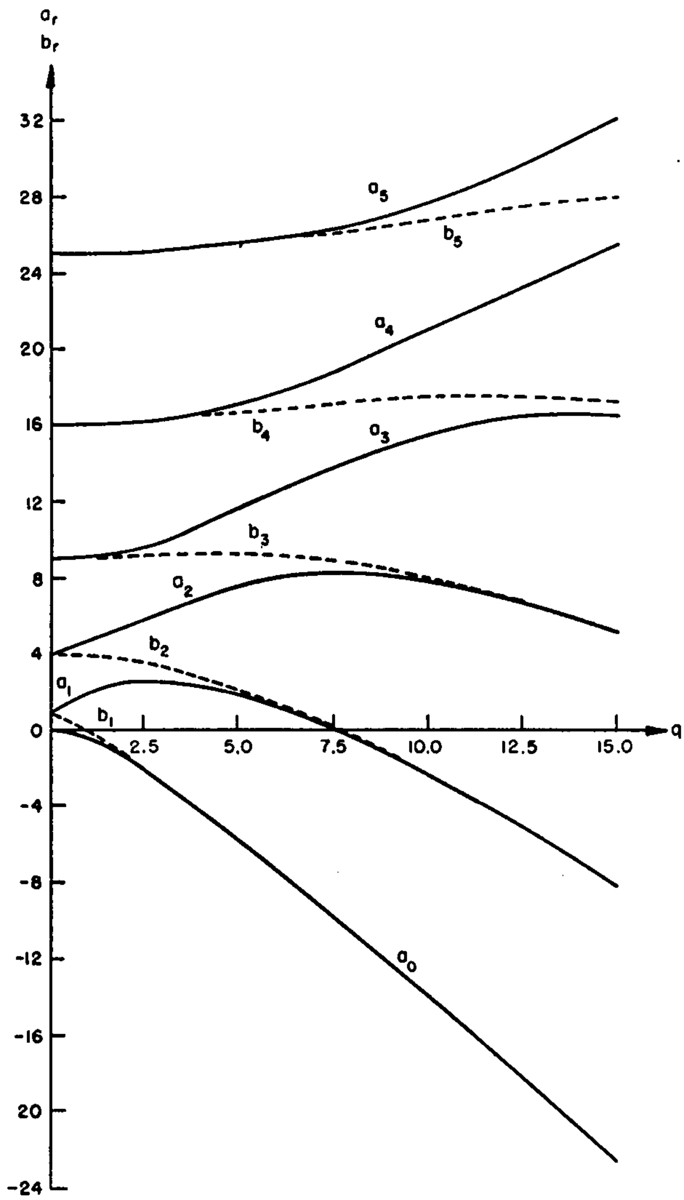

FIGURE 20.1. *Characteristic Values a_r, b_r* $r=0,1(1)5$

Power Series for Characteristic Values

20.2.25

$$a_0(q)=-\frac{q^2}{2}+\frac{7q^4}{128}-\frac{29q^6}{2304}+\frac{68687q^8}{18874368}+\ldots$$

$$\left.\begin{array}{l}a_1(-q)\\ b_1(q)\end{array}\right\}=1-q-\frac{q^2}{8}+\frac{q^3}{64}-\frac{q^4}{1536}-\frac{11q^5}{36864}+\frac{49q^6}{589824}-\frac{55q^7}{9437184}-\frac{83q^8}{35389440}+\ldots$$

$$b_2(q)=4-\frac{q^2}{12}+\frac{5q^4}{13824}-\frac{289q^6}{79626240}+\frac{21391q^8}{458647142400}+\ldots$$

$$a_2(q)=4+\frac{5q^2}{12}-\frac{763q^4}{13824}+\frac{1002401q^6}{79626240}-\frac{1669068401q^8}{458647142400}+\ldots$$

$$\left.\begin{array}{l}a_3(-q)\\ b_3(q)\end{array}\right\}=9+\frac{q^2}{16}-\frac{q^3}{64}+\frac{13q^4}{20480}+\frac{5q^5}{16384}-\frac{1961q^6}{23592960}+\frac{609q^7}{104857600}+\ldots$$

$$b_4(q)=16+\frac{q^2}{30}-\frac{317q^4}{864000}+\frac{10049q^6}{2721600000}+\ldots$$

$$a_4(q)=16+\frac{q^2}{30}+\frac{433q^4}{864000}-\frac{5701q^6}{2721600000}+\ldots$$

$$\left.\begin{array}{l}a_5(-q)\\ b_5(q)\end{array}\right\}=25+\frac{q^2}{48}+\frac{11q^4}{774144}-\frac{q^5}{147456}+\frac{37q^6}{891813888}+\ldots$$

$$b_6(q)=36+\frac{q^2}{70}+\frac{187q^4}{43904000}-\frac{5861633q^6}{92935987200000}+\ldots$$

$$a_6(q)=36+\frac{q^2}{70}+\frac{187q^4}{43904000}+\frac{6743617q^6}{92935987200000}+\ldots$$

For $r\geq 7$, and $|q|$ not too large, a_r is approximately equal to b_r, and the following approximation may be used

20.2.26

$$\left.\begin{array}{l}a_r\\ b_r\end{array}\right\}=r^2+\frac{q^2}{2(r^2-1)}+\frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)}+\frac{(9r^4+58r^2+29)q^6}{64(r^2-1)^5(r^2-4)(r^2-9)}+\ldots$$