20.9.4

$$(m+1)d_{m+1}+\left[\left(m+\frac{1}{2}\right)^2+\left(m+\frac{1}{4}\right)8i\sqrt{q}\,\sigma\right.$$
$$\left.+2q-a\right]d_m+\left(m-\frac{1}{2}\right)[16q(1-\sigma^2)+8i\sqrt{q}\,\sigma m]d_{m-1}$$
$$+4q(2m-3)(2m-1)(1-\sigma^2)d_{m-2}=0.$$

In the above

$$-2\pi<\arg\sqrt{q}\cosh z<\pi$$
$$|\cosh z-\sigma|>|\sigma\pm1|,\ \mathscr{R}z>0,$$

but σ is otherwise arbitrary. If $\sigma^2=1$, **20.9.2** and **20.9.4** become three-term recurrence relations.

Formulas **20.9.1** and **20.9.3** are valid for arbitrary a, q, provided ν is also known; they give multiples of **20.4.12,** normalized so as to approach the corresponding Hankel functions $H_\nu^{(1)}(\sqrt{q}e^z)$, $H_\nu^{(2)}(\sqrt{q}e^z)$, as $z\to\infty$. See [20.36], section **2.63.** The formula is especially useful if $|\cosh z|$ is large and q is not too large; thus if $\sigma=-1$, the absolute ratio of two successive terms in the expansion is essentially

$$\left|\left(\frac{\sqrt{q}}{m}+\frac{m}{4\sqrt{q}}+2\right)/(\cosh z+1)\right|.$$

If a, q, z, ν are real, the real and imaginary components of $Mc_r^{(3)}(z,q)$ are $Mc_r^{(1)}(z,q)$ and $Mc_r^{(2)}(z,q)$, respectively; similarly for the components of $Ms_r^{(3)}(z,q)$. If the parameters are complex

20.9.5 $$Mc_r^{(1)}(z,q)=\tfrac{1}{2}\,[Mc_r^{(3)}(z,q)+Mc_r^{(4)}(z,q)]$$

20.9.6 $$Mc_r^{(2)}(z,q)=-\frac{i}{2}\,[Mc_r^{(3)}(z,q)-Mc_r^{(4)}(z,q)]$$

Replacing c by s in the above will yield corresponding relations among $Ms_r^{(j)}(z,q)$.

Formulas in which the parameter a does not enter explicitly:

Goldstein's Expansions

20.9.7

$$Mc_r^{(3)}(z,q)\sim iMs_{r+1}^{(3)}(z,q)$$
$$\approx[F_0(z)-iF_1(z)]e^{i\phi}/\pi^{\frac{1}{2}}q^{\frac{1}{4}}(\cosh z)^{\frac{1}{2}}$$

where

20.9.8

$$\phi=2\sqrt{q}\sinh z-\frac{1}{2}(2r+1)\arctan\sinh z,$$
$$\mathscr{R}z>0,\ q>>1,\ w=2r+1$$

20.9.9

$$F_0(z)\sim1+\frac{w}{8\sqrt{q}\cosh^2 z}$$
$$+\frac{1}{2048q}\left[\frac{w^4+86w^2+105}{\cosh^4 z}-\frac{w^4+22w^2+57}{\cosh^2 z}\right]$$
$$+\frac{1}{16384q^{3/2}}\left[\frac{-(w^5+14w^3+33w)}{\cosh^2 z}\right.$$
$$\left.-\frac{(2w^5+124w^3+1122w)}{\cosh^4 z}+\frac{3w^5+290w^3+1627w}{\cosh^6 z}\right]+\dots$$

20.9.10

$$F_1(z)\sim\frac{\sinh z}{\cosh^2 z}\left[\frac{w^2+3}{32\sqrt{q}}+\frac{1}{512q}\left(w^3+3w+\frac{4w^3+44w}{\cosh^2 z}\right)\right.$$
$$+\frac{1}{16384q^{3/2}}\left\{5w^4+34w^2+9\right.$$
$$-\frac{(w^6-47w^4+667w^2+2835)}{12\cosh^2 z}$$
$$\left.\left.+\frac{(w^6+505w^4+12139w^2+10395)}{12\cosh^4 z}\right\}\right]+\dots$$

See [20.18] for details and an added term in $q^{-5/2}$; a correction to the latter is noted in [20.58].

The expansions **20.9.7** are especially useful when q is large and z is bounded away from zero. The order of magnitude of $Mc_r^{2}(0,q)$ cannot be obtained from the expansion. The expansion can also be used, with some success, for $z=ix$, when q is large, if $|\cos x|>>0$; they fail at $x=\frac{1}{2}\pi$. Thus, if q, x are real, one obtains

20.9.11

$$ce_r(x,q)\sim\frac{ce_r(0,q)2^{r-\frac{1}{2}}}{F_0(0)}\{W_1[P_0(x)-P_1(x)]$$
$$+W_2[P_0(x)+P_1(x)]\}$$

20.9.12

$$se_{r+1}(x,q)\sim se'_{r+1}(0,q)\tau_{r+1}\{W_1[P_0(x)-P_1(x)]$$
$$-W_2[P_0(x)+P_1(x)]\}$$

In the above, $P_0(x)$ and $P_1(x)$ are obtainable from $F_0(z)$, $F_1(x)$ in **20.9.9–20.9.10** by replacing $\cosh z$ with $\cos x$ and $\sinh z$ with $\sin x$. Thus $P_0(x)=F_0(ix)$; $P_1(x)=-iF_1(ix)$:

20.9.13

$$W_1=e^{2\sqrt{q}\sin x}\,[\cos(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$

$$W_2=e^{-2\sqrt{q}\sin x}\,[\sin(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$