20.9.14

$$\tau_{r+1}\sim 2^{r-\frac{1}{2}}\bigg/\left[2\sqrt{q}-\tfrac{1}{4}w-\frac{(2w^2+3)}{64\sqrt{q}}-\frac{(7w^3+47w)}{1024q}-\ldots\right]$$

See **20.9.23–20.9.24** for expressions relating to $ce_r(0, q)$ and $se_r'(0, q)$. When $|\cos x|>\sqrt{4r+2}/q^{\frac{1}{4}}$, **20.9.11–20.9.12** are useful. The approximations become poorer as r increases.

Expansions in Terms of Parabolic Cylinder Functions

(Good for angles close to $\frac{1}{2}\pi$, for large values of q, especially when $|\cos x|<2^{\frac{1}{2}}/q^{\frac{1}{4}}$.) Due to Sips [20.44–20.46].

20.9.15 $\quad ce_r(x, q)\sim C_r[Z_0(\alpha)+Z_1(\alpha)]$

20.9.16

$$se_{r+1}(x, q)\sim S_r[Z_0(\alpha)-Z_1(\alpha)]\sin x, \qquad \alpha=2q^{\frac{1}{4}}\cos x.$$

Let $D_k=D_k(\alpha)=(-1)^k e^{\frac{1}{4}\alpha^2}\dfrac{d^k}{d\alpha^k}e^{-\frac{1}{4}\alpha^2}$.

20.9.17

$$Z_0(\alpha)\sim D_r+\frac{1}{4q^{\frac{1}{2}}}\left[-\frac{D_{r+4}}{16}+\frac{3}{2}\binom{r}{4}D_{r-4}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+8}}{512}-\frac{(r+2)D_{r+4}}{16}+\frac{3}{2}(r-1)\binom{r}{4}D_{r-4}\right.$$
$$\left.+\frac{315}{4}\binom{r}{8}D_{r-8}\right]+\ldots$$

20.9.18

$$Z_1(\alpha)\sim\frac{1}{4q^{\frac{1}{2}}}\left[-\frac{1}{4}D_{r+2}-\frac{r(r-1)}{4}D_{r-2}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+6}}{64}+\frac{(r^2-25r-36)}{64}D_{r+2}\right.$$
$$\left.+\frac{r(r-1)(-r^2-27r+10)}{64}D_{r-2}-\frac{45}{4}\binom{r}{6}D_{r-6}+\ldots\right]$$

20.9.19

$$C_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1+\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3+263r^2+262r+108}{2048q}+\frac{f_1}{16384q^{\frac{3}{2}}}+\ldots\right]^{-\frac{1}{2}}$$

$$f_1=6r^5+15r^4+1280r^3+1905r^2+1778r+572$$

20.9.20

$$S_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1-\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3-121r^2-122r-84}{2048q}+\frac{f_2}{16384q^{\frac{3}{2}}}+\ldots\right]^{-\frac{1}{2}}$$

$$f_2=2r^5+5r^4-416r^3-629r^2-1162r-476$$

It should be noted that **20.9.15** is also valid as an approximation for $se_{r+1}(x, q)$, but **20.9.16** may give slightly better results. See [20.4.]

Explicit Expansions for Orders 0, 1, to Terms in $q^{-3/2}$ (q Large)

20.9.21 For $r=0$:

$$Z_0\sim D_0-\frac{D_4}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{D_4}{8}+\frac{D_8}{512}\right)\ *$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{99D_4}{256}+\frac{3D_8}{256}-\frac{D_{12}}{24576}\right)+\ldots$$

$$Z_1\sim\frac{-D_2}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{9D_2}{16}+\frac{D_6}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{61D_2}{32}+\frac{25D_6}{256}-\frac{5D_{10}}{10240}\right)+\ldots$$

20.9.22 For $r=1$:

$$Z_0\sim D_1-\frac{D_5}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{3D_5}{16}+\frac{D_9}{512}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{207D_5}{256}+\frac{D_9}{64}-\frac{D_{13}}{24576}\right)+\ldots$$

$$Z_1\sim\frac{-D_3}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{15D_3}{16}+\frac{D_7}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{153D_3}{32}+\frac{35D_7}{256}-\frac{D_{11}}{2048}\right)+\ldots$$

Formulas Involving $ce_r(0, q)$ and $se_r(0, q)$

20.9.23

$$\frac{ce_0(0, q)}{ce_0(\frac{1}{2}\pi, q)}\sim 2\sqrt{2}\,e^{-2\sqrt{q}}\left(1+\frac{1}{16\sqrt{q}}+\frac{9}{256q}+\ldots\right)$$

$$\frac{ce_2(0, q)}{ce_2(\frac{1}{2}\pi, q)}\sim -32q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{1}{16\sqrt{q}}+\frac{29}{128q}+\ldots\right)$$

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*See page II.