**4.4.39**

$$\text{Arctan } z = k\pi + \tfrac{1}{2} \arctan\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{4} \ln\left[\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right] \quad (z^2 \neq -1)$$

where $k$ is an integer or zero and

$$\alpha = \tfrac{1}{2}[(x+1)^2+y^2]^{\frac{1}{2}} + \tfrac{1}{2}[(x-1)^2+y^2]^{\frac{1}{2}}$$

$$\beta = \tfrac{1}{2}[(x+1)^2+y^2]^{\frac{1}{2}} - \tfrac{1}{2}[(x-1)^2+y^2]^{\frac{1}{2}}$$

### Series Expansions

**4.4.40**

$$\arcsin z = z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3 z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5 z^7}{2\cdot 4\cdot 6\cdot 7} + \ldots \quad (|z|<1)$$

**4.4.41**

$$\arcsin (1-z) = \frac{\pi}{2} - (2z)^{\frac{1}{2}}\left[1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5 \ldots (2k-1)}{2^{2k}(2k+1)k!} z^k\right] \quad (|z|<2)$$

**4.4.42**

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots \quad (|z| \leq 1 \text{ and } z^2 \neq -1)$$

$$= \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \ldots \quad (|z|>1 \text{ and } z^2 \neq -1)$$

$$= \frac{z}{1+z^2}\left[1 + \frac{2}{3}\frac{z^2}{1+z^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{z^2}{1+z^2}\right)^2 + \ldots\right] \quad (z^2 \neq -1)$$

### Continued Fractions

**4.4.43** $$\arctan z = \frac{z}{1+}\,\frac{z^2}{3+}\,\frac{4z^2}{5+}\,\frac{9z^2}{7+}\,\frac{16z^2}{9+} \ldots$$

($z$ in the cut plane of Figure 4.4.)

**4.4.44** $$\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{z}{1-}\,\frac{1\cdot 2z^2}{3-}\,\frac{1\cdot 2z^2}{5-}\,\frac{3\cdot 4z^2}{7-}\,\frac{3\cdot 4z^2}{9-} \ldots$$

($z$ in the cut plane of Figure 4.4.)

### Polynomial Approximations [9]

**4.4.45** $0 \leq x \leq 1$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-5}$$

| | |
|---|---|
| $a_0 = 1.57072\ 88$ | $a_2 = .07426\ 10$ |
| $a_1 = -.21211\ 44$ | $a_3 = -.01872\ 93$ |

[9] The approximations **4.4.45** to **4.4.47** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

**4.4.46** $0 \leq x \leq 1$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7) + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

| | |
|---|---|
| $a_0 = 1.57079\ 63050$ | $a_4 = .03089\ 18810$ |
| $a_1 = -.21459\ 88016$ | $a_5 = -.01708\ 81256$ |
| $a_2 = .08897\ 89874$ | $a_6 = .00667\ 00901$ |
| $a_3 = -.05017\ 43046$ | $a_7 = -.00126\ 24911$ |

**4.4.47** $-1 \leq x \leq 1$

$$\arctan x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \epsilon(x)$$

$$|\epsilon(x)| \leq 10^{-5}$$

| | |
|---|---|
| $a_1 = .99986\ 60$ | $a_7 = -.08513\ 30$ |
| $a_3 = -.33029\ 95$ | $a_9 = .02083\ 51$ |
| $a_5 = .18014\ 10$ | |

**4.4.48** [10] $-1 \leq x \leq 1$

$$\arctan x = \frac{x}{1 + .28x^2} + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-3}$$

**4.4.49** [11] $0 \leq x \leq 1$

$$\frac{\arctan x}{x} = 1 + \sum_{k=1}^{8} a_{2k} x^{2k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

| | |
|---|---|
| $a_2 = -.33333\ 14528$ | $a_{10} = -.07528\ 96400$ |
| $a_4 = .19993\ 55085$ | $a_{12} = .04290\ 96138$ |
| $a_6 = -.14208\ 89944$ | $a_{14} = -.01616\ 57367$ |
| $a_8 = .10656\ 26393$ | $a_{16} = .00286\ 62257$ |

[10] The approximation **4.4.48** is from C. Hastings, Jr., Note 143, Math. Tables Aids Comp. **6**, 68 (1953) (with permission).

[11] The approximation **4.4.49** is from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA-1943, Los Alamos, N. Mex., 1955 (with permission).