Inequalities for distribution functions

($F(x)$ denotes the c.d.f. of the random variable X and t denotes a positive constant; further m is always assumed to be finite and all expectations are assumed to exist.)

Inequality	Conditions
26.1.35 $Pr\{g(X)\geq t\}\leq E[g(X)]/t$	(i) $g(X)\geq 0$
26.1.36 $Pr\{X\geq t\}\leq m/t$ $F(t)\geq 1-\frac{m}{t}$	(i) $Pr\{X<0\}=0$ (ii) $E(X)=m$
26.1.37 $Pr\{\|X-m\|\geq t\sigma\}\leq 1/t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{1}{t^2}$	(i) $E(X)=m$ * (ii) $E(X-m)^2=\sigma^2$
26.1.38 $Pr\{\|\overline{X}-\overline{m}\|\geq t\overline{\sigma}\}\leq\frac{1}{nt^2}$	(i) $E(X_i)=m_i$ (ii) $E(X_i-m_i)^2=\sigma_i^2$ (iii) $E([X_i-m_i][X_j-m_j])=0\ (i\neq j)$ (iv) $\overline{X}=\sum_{i=1}^{n}\frac{X_i}{n}$, $\overline{m}=\sum_{i=1}^{n}\frac{m_i}{n}$, $\overline{\sigma}=\left[\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right]^{\frac{1}{2}}$
26.1.39 $Pr\{\|X-m\|\geq t\sigma\}\leq\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\|\frac{m-x_0}{\sigma}\right\|\right)^2}\right\}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\|\frac{m-x_0}{\sigma}\right\|\right)^2}\right\}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6]
26.1.40 $Pr\{\|X-m\|\geq t\sigma\}\leq 4/9t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9t^2}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6] (iv) $m=x_0$
26.1.41 $Pr\{\|X-m\|\geq t\sigma\}\leq\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$	(i) $E(X-m)^2=\sigma^2$ (ii) $E(X-m)^4=\mu_4$

[6] x_0 is such that $F'(x_0)>F'(x)$ for $x\neq x_0$.

26.2. Normal or Gaussian Probability Function

26.2.1 $$Z(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

26.2.2 $$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt=\int_{-\infty}^{x}Z(t)dt$$

26.2.3 $$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-t^2/2}dt=\int_{x}^{\infty}Z(t)dt$$

26.2.4 $$A(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t^2/2}dt=\int_{-x}^{x}Z(t)dt$$

26.2.5 $$P(x)+Q(x)=1$$

26.2.6 $$P(-x)=Q(x)$$

26.2.7 $$A(x)=2P(x)-1$$

Probability Integral with Mean m and Variance σ^2

A random variable X is said to be normally distributed with mean m and variance σ^2 if the probability that X is less than or equal to x is given by

26.2.8
$$Pr\{X\leq x\}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-m)^2}{2\sigma^2}}dt$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-m)/\sigma}e^{-t^2/2}dt=P\left(\frac{x-m}{\sigma}\right).$$

The corresponding probability density function is

26.2.9
$$\frac{\partial}{\partial x}P\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma}Z\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$$

and is symmetric around m, i.e.

$$Z\left(\frac{m+x}{\sigma}\right)=Z\left(\frac{m-x}{\sigma}\right).$$

The inflexion points of the probability density function are at $m\pm\sigma$.