Integral Over an Ellipse With Center at (m_x, m_y)

26.3.21

$$\iint_A (\sigma_x\sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dxdy = 1 - e^{-a^2/2}$$

where A is the area enclosed by the ellipse

$$\left(\frac{x-m_x}{\sigma_x}\right)^2 - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \left(\frac{y-m_y}{\sigma_y}\right)^2 = a^2(1-\rho^2)$$

Integral Over an Arbitrary Region

26.3.22

$$\iint_{A(x,y)} (\sigma_x\sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dxdy = \iint_{A^*(s,t)} g(s, t, o) dsdt$$

where $A^*(s, t)$ is the transformed region obtained from the transformation

$$s = \frac{1}{\sqrt{2+2\rho}}\left(\frac{x-m_x}{\sigma_x} + \frac{y-m_y}{\sigma_y}\right)$$

$$t = \frac{-1}{\sqrt{2-2\rho}}\left(\frac{x-m_x}{\sigma_x} - \frac{y-m_y}{\sigma_y}\right)$$

Integral of the Circular Normal Probability Function With Parameters $m_x = m_y = 0$, $\sigma = 1$ Over the Triangle Bounded by $y=0$, $y=ax$, $x=h$

26.3.23

$$V(h, ah) = \frac{1}{2\pi}\int_0^h \int_0^{ax} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

$$= \frac{1}{4} + L(h, 0, \rho) - L(0, 0, \rho) - \frac{1}{2} Q(h)$$

where

$$\rho = -\frac{a}{\sqrt{1+a^2}}$$

Integral of Circular Normal Distribution Over an Offset Circle With Radius $R\sigma$ and Center a Distance $r\sigma$ From (m_x, m_y)

26.3.24

$$\int_A \int \sigma^{-2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) dxdy = P(R^2|2, r^2)$$

where $P(R^2|2, r^2)$ is the c.d.f. of the non-central χ^2 distribution (see **26.4.25**) with $\nu=2$ degrees of freedom and noncentrality parameter r^2.

Approximation to $P(R^2|2, r^2)$

	Approximation	*Condition*
26.3.25	$\frac{2R^2}{4+R^2} \exp - \frac{2r^2}{4+R^2}$	$R<1$
26.3.26	$P(x_1)$	$R>1$
26.3.27	$P(x_2)$	$R>5$

$$x_1 = \frac{[R^2/(2+r^2)]^{1/3} - \left[1 - \frac{2}{9}\frac{2+2r^2}{(2+r^2)^2}\right]}{\left[\frac{2}{9}\frac{2+2r^2}{(2+r^2)^2}\right]^{\frac{1}{2}}}$$

$x_2 = R - \sqrt{r^2-1}$ $\quad$ R, r both large *

Inequality

26.3.28

$$Q(h) - \frac{1-\rho^2}{\rho h - k} Z(k)\left[Q\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] < L(h, k, \rho) < Q(h)$$

where

$$\rho h - k > 0, \qquad 0 < \rho < 1.$$

Series Expansion

26.3.29

$$L(h, k, \rho) = Q(h)Q(k) + \sum_{n=0}^{\infty} \frac{Z^{(n)}(h)Z^{(n)}(k)}{(n+1)!} \rho^{n+1}$$

26.4. Chi-Square Probability Function

26.4.1

$$P(\chi^2|\nu) = \left[2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1} \int_0^{\chi^2} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \qquad (0 \le \chi^2 < \infty)$$

26.4.2

$$Q(\chi^2|\nu) = 1 - P(\chi^2|\nu) \qquad (0 \le \chi^2 < \infty)$$

$$= \left[2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1} \int_{\chi^2}^{\infty} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt$$

Relation to Normal Distribution

Let $X_1, X_2, \ldots, X_\nu$ be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then $X^2 = \sum_{i=1}^{\nu} X_i^2$ is said to follow the chi-square distribution with ν degrees of freedom and the probability that $X^2 \le \chi^2$ is given by $P(\chi^2|\nu)$.

Cumulants

26.4.3 $\quad \kappa_{n+1} = 2^n n! \nu \qquad (n=0, 1, \ldots)$

*See page II.