It is of interest to note that the standard normal distribution is unimodal, has mean zero, unit variance $\mu_4=3$, is continuous, and such that

$$A(t)=P(t)-P(-t) = .6827,\ .9545,\ .9973, \text{ and } .9999$$

for $t=1$, 2, 3 and 4 respectively.

Interpolation for $P(x)$ in Table 26.1

Example 2. Compute $P(x)$ for $x=2.576$ to fifteen decimal places using a Taylor expansion.

Writing $x=x_0+\theta$ we have

$$P(x)=P(x_0)+Z(x_0)\theta+Z^{(1)}(x_0)\frac{\theta^2}{2!}+Z^{(2)}(x_0)\frac{\theta^3}{3!}+Z^{(3)}(x_0)\frac{\theta^4}{4!}+\ldots$$

Taking $x_0=2.58$ and $\theta=-4\times10^{-3}$ we calculate the successive terms to 16D

+.99505	99842	42230		
−	5	72204	35976	6
−		2952	57449	6
−		8	63097	8
−			1439	4
−				9
.99500	24676	84265	7	

The result correct to 17D is

$$P(2.576)=.99500\ 24676\ 84264\ 98$$

Calculation for Arbitrary Mean and Variance

Example 3. Find the value to 5D of

$$P\{X\leq .50\}=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{.5}e^{-1/2\left(\frac{t-1}{2}\right)^2}dt$$

using **26.2.8** and **Table 26.1.**

This represents the probability of the random variable being less than or equal to .5 for a normal distribution with mean $m=1$ and variance $\sigma^2=4$. Using **26.2.8** we have

$$P\{X\leq .5\}=P\left(\frac{.5-1}{2}\right)=P(-.25)$$

Since $P(-x)=1-P(x)$, we have

$$P(-.25)=1-P(.25)=1-.59871=.40129$$

where a two-term Taylor series was used for interpolation. Note that when interpolating for $P(x)$ for a value of x midway between the tabulated values we can write $x=x_0+.01$ and a two-term Taylor series is $P(x)=P(x_0)+Z(x_0)10^{-2}$. Thus one need only multiply $Z(x_0)$ by 10^{-2} and add the result to $P(x_0)$.

Calculation of $P(x)$ for x Approximate

Example 4. Using **Table 26.1,** find $P(x)$ for $x=1.96$, when there is a possible error in x of $\pm5\times10^{-3}$.

This is an example where the argument is only known approximately. The question arises as to how many decimal places one should retain in $P(x)$. If Δx and $\Delta P(x)$ denote the error in x and the resulting error in $P(x)$, respectively, then

$$\Delta P(x)\approx Z(x)\Delta x$$

Hence $\Delta P(1.960)=3\times10^{-4}$ which indicates that $P(1.960)$ need only be calculated to 4D. Therefore $P(1.960)=.9750$.

Inverse Interpolation for $P(x)$

Example 5. Find the value of x for which $P(x)=.97500\ 00000\ 00000$ using **Table 26.1** and determining as many decimal places as is consistent with the tabulated function.

For inverse interpolation the tabulated function $P(x)$ may be regarded as having a possible error of $.5\times10^{-15}$. Hence

$$\Delta x\approx\frac{\Delta P(x)}{Z(x)}=\frac{.5\times10^{-15}}{Z(x)}$$

Let $P(x_0)$ correspond to the closest tabulated value of $P(x)$. Then a convenient formula for inverse interpolation is

$$x=x_0+t+\frac{x_0t^2}{2}+\frac{2x_0^2+1}{6}t^3$$

where

$$t=\frac{P(x)-P(x_0)}{Z(x_0)}$$

If only the first two terms (i.e., $x=x_0+t$) are used, the error in x will be bounded by $\frac{x}{8}\times10^{-4}$ and the true value will always be greater than the value thus calculated.

With respect to this example, $\Delta x\approx10^{-14}$ and thus the interpolated value of x may be in error by one unit in the fourteenth place. The closest value to $P(x)=.97500\ 00000\ 00000$ is $P(x_0)=.97500\ 21048\ 51780$ with $x_0=1.96$. Hence using the preceding inverse interpolation formulas with